

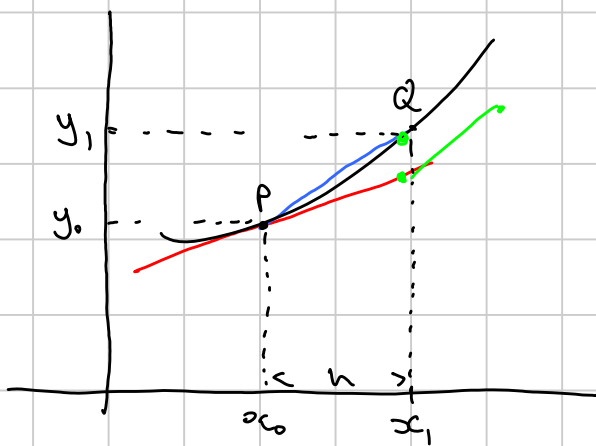
Chapter 5 - Step-by-step methods for DEs

Note Title

13/10/2008

Given a DE and some initial conditions (x_0, y_0) which lie on the curve, these methods generate a sequence of points (x_1, y_1) , (x_2, y_2) ... which lie approximately on the curve. The further we go from the initial conditions, the less reliable the approximation.

First Order DEs



Given a DE which defines a curve, a point (x_0, y_0) on the curve and a step value h , we wish to find an approximation for y_1 (which is $f(x_1)$) at the point x_1 (which is $x_0 + h$)

Now the gradient of PQ can be approximated by the gradient of the curve at P , which can be written as $f'(x_0)$ or $\left(\frac{dy}{dx}\right)_0$

$$\text{i.e.} \quad \boxed{\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h}}$$

* These will be given when required but their derivation could be asked for.

Rearranging gives

$$y_1 \approx y_0 + h \left(\frac{dy}{dx}\right)_0$$

or, in a form suitable for iteration:

$$y_{n+1} \approx y_n + h \left(\frac{dy}{dx} \right)_n$$

Example Given the DE $\frac{dy}{dx} = \frac{2x-y}{x+1}$, with $y=0.5$

when $x=1$, use the above method to find the approximate value of y at $x=1.2, 1.4$ and 1.6

We have

$$x_0 = 1, \quad y_0 = 0.5, \quad h = 0.2$$

$$\text{So } \left(\frac{dy}{dx} \right)_0 = \frac{2 - 0.5}{1+1} = 0.75$$

$$\text{So } x_1 = 1.2, \quad y_1 = 0.5 + 0.2 \times 0.75 \\ = 0.65$$

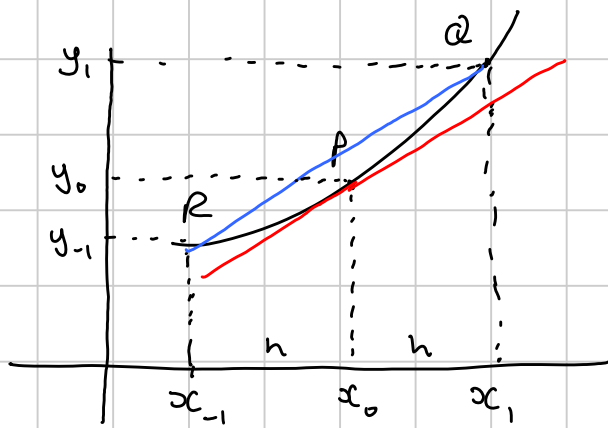
$$\left(\frac{dy}{dx} \right)_1 = \frac{2 \times 1.2 - 0.65}{1.2+1} = 0.795 \dots$$

$$\text{So } x_2 = 1.4, \quad y_2 = 0.65 + 0.2 \times 0.795 \dots \\ = 0.809 \dots$$

$$\left(\frac{dy}{dx} \right)_2 = \frac{2 \times 1.4 - 0.809 \dots}{1.4+1} = 0.829 \dots$$

$$\text{So } x_3 = 1.6, \quad y_3 = 0.809 \dots + 0.2 \times 0.829 \dots \\ = \underline{\underline{0.975}}$$

The above method can be improved as follows:—



Gradient at $P \approx$ Gradient of RQ

$$\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_{-1}}{2h} *$$

Rearranging, $y_1 \approx y_{-1} + 2h \left(\frac{dy}{dx}\right)_0$

or as an iterative formula

$$y_{n+1} \approx y_{n-1} + 2h \left(\frac{dy}{dx}\right)_n$$

To use this we need to know two points. So we need to use the previous method once to get (x_1, y_1) - then we can switch to this method.

Example Apply this 'improved' method to the previous example. $\left(\frac{dy}{dx} = \frac{2x - y}{x + 1} \text{ with } y = 0.5 \text{ when } x = 1\right)$

We have $x_0 = 1$, $y_0 = 0.5$

and we use the previous method once to find $x_1 = 1.2$, $y_1 = 0.65$

Now $x_2 = 1.4$, and $y_2 \approx y_0 + 2h \left(\frac{dy}{dx}\right)_1$
 $\approx 0.5 + 0.4 \times 0.795 \dots$

$$\approx 0.8181\dots$$

$$\text{So } \left(\frac{dy}{dx}\right)_2 = \frac{2 \times 1.4 - 0.8181\dots}{1.4 + 1} = 1.09\dots$$

$$\begin{aligned} \text{And } x_3 = 1.6, \quad y_3 &\approx y_1 + 2h \left(\frac{dy}{dx}\right)_2 \\ &\approx 0.65 + 0.4 \times 0.8257\dots \\ &\approx \underline{\underline{0.980}} \end{aligned}$$

[To solve this DE exactly,

$$\frac{dy}{dx} = \frac{2x - y}{x + 1}$$

$$\frac{dy}{dx} + \frac{1}{x+1} y = \frac{2x}{x+1}$$

$$\text{IF} = e^{\int \frac{1}{x+1} dx} = e^{\ln(x+1)} = x+1$$

$$(x+1) \frac{dy}{dx} + y = 2x$$

$$\frac{d}{dx} (y(x+1)) = 2x$$

$$y(x+1) = x^2 + C$$

$$y = \frac{x^2 + C}{x+1}$$

$$\begin{aligned} \text{When } x=1, y=0.5 &\Rightarrow 0.5 = \frac{1+C}{2} \\ &\Rightarrow C = 0 \end{aligned}$$

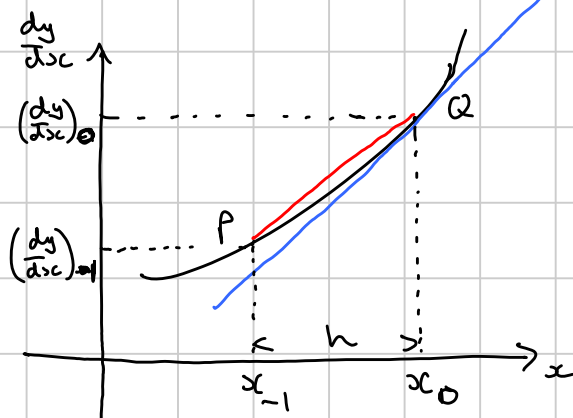
$$\text{Hence } y = \frac{x^2}{x+1}$$

$$\begin{aligned} \text{When } x=1.6, y &= \frac{2.56}{2.6} \\ &= \underline{\underline{0.985}} \end{aligned}$$

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Second Order DEs

$\frac{d^2y}{dx^2}$ is the gradient of $\frac{dy}{dx}$



gradient of tangent at Q \approx gradient of chord PQ.

$$\left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{\left(\frac{dy}{dx}\right)_0 - \left(\frac{dy}{dx}\right)_{-1}}{h}$$

(use the first form of approximation above)

$$\approx \frac{\frac{y_1 - y_0}{h} - \frac{y_0 - y_{-1}}{h}}{h}$$

$$\left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2} *$$

Examples

① Given the DE $\frac{d^2y}{dx^2} = x - y$, with $y=1$ and $\frac{dy}{dx}=2$

when $x=0$, find the approximate value of y at $x=0.1$
and $x=0.2$

We know $h=0.1$

$$x_0 = 0, \quad y_0 = 1, \quad \left(\frac{dy}{dx}\right)_0 = 2$$

$$\text{We can find } \left(\frac{d^2y}{dx^2}\right)_0 = 0 - 1 = -1$$

So in the formula above,

$$-1 = \frac{y_1 - 2 + y_{-1}}{0.01} \quad \text{--- (1)}$$

But also, using $\left(\frac{dy}{dx}\right)_0 = \frac{y_1 - y_{-1}}{2h}$

$$2 = \frac{y_1 - y_{-1}}{0.2} \quad \text{--- (2)}$$

$$\left. \begin{array}{l} \text{(1)} \Rightarrow y_1 + y_{-1} = 1.99 \\ \text{(2)} \Rightarrow y_1 - y_{-1} = 0.4 \end{array} \right\}$$

$$\text{So } y_1 = \frac{2.39}{2} = \underline{\underline{1.195}} \quad (\text{at } x_1 = 0.1)$$

Now we can use $\left(\frac{d^2y}{dx^2}\right)_1 = \frac{y_2 - 2y_1 + y_0}{h^2}$

$$\begin{aligned} \text{and } \left(\frac{d^2y}{dx^2}\right)_1 &= x_1 - y_1 \\ &= 0.1 - 1.195 \\ &= -1.095 \end{aligned}$$

$$\text{So } -1.095 = \frac{y_2 - 2 \times 1.195 + 1}{0.01}$$

$$\begin{aligned} y_2 &= 2.39 - 1 - 0.01095 \\ &= \underline{\underline{1.37905}} \end{aligned}$$

(see notes on Series approximations, which gave $y = 1.379$ at $x = 0.2$)

(2) Given $\frac{d^2y}{dx^2} - \frac{1}{2}y \frac{dy}{dx} - 1 = 0$, and

$y = 2$ when $x = 0$, $y = 2.1$ when $x = 0.1$,
find y when $x = 0.2$.

We have $x_{-1} = 0$, $y_{-1} = 2$ $h = 0.1$
 $x_0 = 0.1$, $y_0 = 2.1$

$$\left(\frac{d^2y}{dx^2}\right)_0 = \frac{y_1 - 2 \times 2.1 + 2}{0.01}$$

$$\left(\frac{dy}{dx}\right)_0 = \frac{y_1 - 2}{0.2} \quad (\text{using the "better" approximation})$$

Sub into DE (using subscript 0 throughout)

$$\frac{y_1 - 2.2}{0.01} - \frac{1}{2}(2.1)\left(\frac{y_1 - 2}{0.2}\right) - 1 = 0$$

$$100y_1 - 220 - \frac{5}{2} \times 2.1(y_1 - 2) - 1 = 0$$

$$200y_1 - 440 - 10.5y_1 + 21 - 2 = 0$$

$$189.5y_1 = 421$$

$$\underline{y_1 = 2.222} \quad (3 \text{ dp})$$

(at $x_1 = 0.2$)

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[Example 1 again: -

$$\frac{d^2y}{dx^2} + y = x$$

AE $x^2 + 1 = 0$
 $x = \pm i$

CF $y = R \cos x + S \sin x$

Trial PI $y = ax + b \Rightarrow \frac{dy}{dx} = a, \frac{d^2y}{dx^2} = 0$

$$0 + ax + b = x \Rightarrow a = 1, b = 0$$

General solution $y = R \cos x + S \sin x + x$

$$\frac{dy}{dx} = S \cos x - R \sin x + 1.$$

$$y = 1 \text{ when } x = 0 \Rightarrow 1 = R$$

$$\frac{dy}{dx} = 2 \text{ when } x = 0 \Rightarrow 2 = S + 1 \Rightarrow S = 1$$

So

$$\underline{\underline{y = \cos x + \sin x + x.}}$$

When $x = 0.2$,

$$y = \cos 0.2 + \sin 0.2 + 0.2.$$

$$= 1.3787 \dots$$

$$= \underline{\underline{1.379}} \quad (3 \text{ dp})$$

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