Differential Equations

(a) First Order Linear D.Es

The general equation of this type is

$$p(x)\frac{dy}{dx} + q(x)\,y = r(x)$$

It is usually convenient to divide through by $p(x)$ to give

$$\frac{dy}{dx} + f(x)\,y = g(x)$$

Examples

① $$x^3\frac{dy}{dx} + 3x^2 y = \cos x$$

By the product rule, the LHS is the derivative of $x^3 y$.

ie/ $$\frac{d}{dx}\left(x^3 y\right) = \cos x$$

Integrate both sides wrt x:

$$x^3 y = \int \cos x \, dx$$

$$\underline{\underline{x^3 y = \sin x + c}}$$

An equation like this where $p'(x) = q(x)$ is called an EXACT equation and is a fluke!
We would like to find a way of converting any equation of this type into an EXACT equation

Starting from

$$\frac{dy}{dx} + f(x)\,y = g(x)$$

we want to find a function $m(x)$ which we can multiply through by so that

$$m(x)\frac{dy}{dx} + m(x)f(x)\,y = m(x)\,g(x)$$

is an exact equation.

ie, we want $\quad m'(x) = m(x)f(x)$

or $\quad \dfrac{dm}{dx} = m(x)f(x)$

Integrating by separating variables

$$\int \frac{1}{m}\,dm = \int f(x)\,dx$$

$$\ln m = \int f(x)\,dx$$

$$\boxed{m = e^{\int f(x)\,dx}}$$

This m is called the INTEGRATING FACTOR for the DE.

<u>Examples</u>

(1) $$\frac{dy}{dx} + \frac{3y}{x} = x^2$$

Here $f(x) = \frac{3}{x}$, so the I.F. is

$$e^{\int \frac{3}{x}\,dx}$$

$$= e^{3\ln x} = x^3$$

Multiplying by this gives

$$x^3 \frac{dy}{dx} + x^3 \frac{3}{x} y = x^5$$

which is an exact equation

$$\frac{d}{dx}\left(x^3 y\right) = x^5$$

$$x^3 y = \frac{1}{6} x^6 + C$$

$$\underline{\underline{y = \frac{1}{6} x^3 + C x^{-3}}}$$

(2) $$\cos x \frac{dy}{dx} + \sin x \, y = 1$$

Divide by $\cos x$:

$$\frac{dy}{dx} + \tan x \, y = \sec x$$

I.F. is $e^{\int \tan x \, dx}$

$$= e^{\ln(\sec x)}$$

$$= \sec x$$

D.E becomes

$$\sec x \frac{dy}{dx} + \tan x \sec x \, y = \sec^2 x$$

[Check: $\frac{d}{dx}(\sec x) = \tan x \sec x$]

$$\frac{d}{dx}\left(\sec x \, y\right) = \sec^2 x$$

$$\sec x \, y = \int \sec^2 x \, dx$$

$$= \tan x + c$$

$$y = \frac{\tan x}{\sec x} + \frac{c}{\sec x}$$

$= \sin x + c\cos x$

[Check: $\frac{dy}{dx} = \cos x - c\sin x$

Subst into DE:

$LHS = \cos x(\cos x - c\sin x) + \sin x(\sin x + c\cos x)$

$= \cos^2 x + \sin^2 x$

$= 1 = RHS$]

p 90 Ex 5C Q 3, 6, 8, 10, 13, 16

(b) Second Order D.E.s

We are going to consider 2nd order LINEAR DEs with constant coefficients :-

$$A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy = f(x)$$

First we consider the case where the RHS is 0.

A possible solution to this is $y = e^{\alpha x}$ for some value of α. Trying this gives

$$A\alpha^2 e^{\alpha x} + B\alpha e^{\alpha x} + Ce^{\alpha x} = 0$$

For this to be true we must have

$$\boxed{A\alpha^2 + B\alpha + C = 0}$$

This quadratic is called the Auxiliary Equation (AE) of the DE

Suppose this has roots α_1 and α_2; then

$$y = e^{\alpha_1 x} \quad \text{and} \quad y = e^{\alpha_2 x}$$

are solutions of the DE.

Furthermore, any LINEAR COMBINATION of these solutions will also be a solution. So the most general form of the solution is

$$\boxed{y = Pe^{\alpha_1 x} + Qe^{\alpha_2 x}}$$

__Example__ $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

AE : $\alpha^2 + \alpha - 6 = 0$

$(\alpha + 3)(\alpha - 2) = 0$

$\alpha = -3$ or $\alpha = 2$.

General solution : $y = Pe^{-3x} + Qe^{2x}$

There are two situations where the above breaks down:

(i) the roots are complex ($B^2 - 4AC < 0$)

(ii) the roots are equal ($B^2 - 4AC = 0$)

(i) In this case the roots are complex conjugates:

$$\boxed{\alpha_1 = u + vi \quad \text{and} \quad \alpha_2 = u - vi}$$

So the general solution is

$$y = Pe^{(u+vi)x} + Qe^{(u-vi)x}$$

$$y = e^{ux}(Pe^{vix} + Qe^{-vix})$$

$$\boxed{y = e^{ux}(R\cos vx + S\sin vx)}$$

(wait for FP3 to prove the last line!)

__Example__ $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 0$

AE : $\alpha^2 - 2\alpha + 4 = 0$

$(\alpha - 1)^2 + 3 = 0$

$$(\alpha-1)^2 = -3$$
$$\alpha - 1 = \pm i\sqrt{3}$$
$$\alpha = 1 \pm i\sqrt{3}$$

General Solution:

$$y = e^{x}\left(R\cos(\sqrt{3}\,x) + S\sin(\sqrt{3}\,x)\right)$$

[Note that by a trig identity of the "$R\sin(\theta+\alpha)$" type, we can write this as

$$y = Te^{ux}\cos(vx + \epsilon)$$

which is an equation of damped or forced harmonic motion (depending on the value of u)]

(ii) If the roots are equal, α_1 and α_1 the solution seems to be

$$y = Pe^{\alpha_1 x} + Qe^{\alpha_1 x}$$

But this is just $y = (P+Q)e^{\alpha_1 x}$

ie, $y = Re^{\alpha_1 x}$ with only one arbitrary constant.

It turns out that

$$y = Qxe^{\alpha_1 x}$$

is also a valid solution in this case.

Thus the general solution in this case is

$$\boxed{y = Pe^{\alpha_1 x} + Qxe^{\alpha_1 x}}$$

__Example__ $$\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25 = 0$$

AE $$\alpha^2 - 10\alpha + 25 = 0$$
$$(\alpha - 5)^2 = 0$$
$$\alpha = 5 \text{ or } 5$$

General Solution :
$$y = Pe^{5x} + Qxe^{5x}$$
or $$y = e^{5x}(P + Qx)$$

p 96 Ex 6A 7, 8
6B 8, 9
6C 8, 9

Now we need to consider the case where $f(x)$ is not 0.

ie/ $$A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy = f(x) \quad (*)$$

To solve eqns of this type we use the fact that if $y = g(x)$ satisfies the given eqn with 0 on the RHS
and $y = h(x)$ satisfies the given eqn,
then $y = g(x) + h(x)$ also satisfies the given eqn.

So our method is

1) Solve the given eqn with RHS = 0

(as above). This is called the COMPLEMENTARY FUNCTION (CF)

2) Solve the given eqn, by "trying" a suitable function. This is called the PARTICULAR INTEGRAL (PI)

3) Then the GENERAL SOLUTION = CF + PI

The form of the "particular integral" to try must include $f(x)$ and all terms which may arise in any derivatives of $f(x)$.

e.g.

RHS	Trial PI
e^{ax}	ke^{ax}
$p \sin ax$ $p \cos ax$ $p \sin ax + q \cos ax$	$c \sin ax + d \cos ax$
ax $ax + b$	$cx + d$
ax^2 $ax^2 + bx + c$	$cx^2 + dx + e$

<u>Example</u> Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = \sin x$

given that when $x = 0$, $y = 0$ and $\frac{dy}{dx} = 0$

AE: $\alpha^2 - 2\alpha - 8 = 0$

$(\alpha - 4)(\alpha + 2) = 0$

CF: $y = Pe^{4x} + Qe^{-2x}$

Trial PI: $y = c\sin x + d\cos x$

$$\frac{dy}{dx} = c\cos x - d\sin x$$

$$\frac{d^2y}{dx^2} = -c\sin x - d\cos x$$

Subst in DE:

$$(-c\sin x - d\cos x) - 2(c\cos x - d\sin x) - 8(c\sin x + d\cos x) = \sin x$$

$$-c + 2d - 8c = 1 \quad ①$$

$$-d - 2c - 8d = 0 \quad ②$$

① × 9 ⇒ $18d - 81c = 9$

② × 2 ⇒ $-18d - 4c = 0$

$$-85c = 9$$

$$c = -9/85$$

① ⇒ $2d + 81/85 = 1$

$$d = 2/85$$

PI is $y = \frac{1}{85}(2\cos x - 9\sin x)$

General solution of the DE:

$$y = Pe^{4x} + Qe^{-2x} + \frac{1}{85}(2\cos x - 9\sin x)$$

Initial conditions :-

when $x = 0$, $y = 0$

$$\Rightarrow \quad 0 = P + Q + \frac{2}{85} \qquad ①$$

$$\frac{dy}{dx} = 4Pe^{4x} - 2Qe^{-2x} + \frac{1}{85}(-2\sin x - 9\cos x)$$

when $x = 0$, $\frac{dy}{dx} = 0$

$$\Rightarrow \quad 0 = 4P - 2Q - \frac{9}{85} \qquad ②$$

$$① \times 2 + ② \Rightarrow 0 = 6P - \frac{5}{85}$$

$$P = \frac{1}{102}$$

$$① \times 4 - ② \Rightarrow 0 = 6Q + \frac{17}{85}$$

$$Q = -\frac{1}{30}$$

Solution is :

$$y = \frac{1}{102}e^{4x} - \frac{1}{30}e^{-2x} + \frac{1}{85}(2\cos x - 9\sin x)$$

p105 Ex 6D Q 1, 2, 3, 5, 17

The 'failure case'

If the trial PI already forms part of the CF, we will be unable to find values of the constants to give a correct PI, because any values of the constants will make the RHS equal to 0 rather than f(x).

eg. Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 3e^{2x}$

AE: $\alpha^2 - 5\alpha + 6 = 0$

$(\alpha - 2)(\alpha - 3) = 0$

CF: $y = Pe^{2x} + Qe^{3x}$

Trial PI: $y = ke^{2x}$

$\frac{dy}{dx} = 2ke^{2x}$

$\frac{d^2y}{dx^2} = 4ke^{2x}$

subst: $4ke^{2x} - 10ke^{2x} + 6ke^{2x} = 3e^{2x}$

compare coeff: $0 = 3$

doesn't work.

In this case we multiply the trial PI by x.
So for the above example we would take

Trial PI: $y = kxe^{2x}$

If this is still in the CF, we need to multiply by x again.

e.g. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 3e^{2x}$

CF $y = Pe^{2x} + Qxe^{2x}$

So Trial PI: $y = kx^2e^{2x}$

p 105 Ex 6D Q 10, 11, 27

(c) Solving Differential Eqns by Substitution

There are many DEs which can be reduced by a suitable substitution to one of the types already seen:

- separating variables
- 'integrating factor'
- CF + PI

The necessary substitution will be given in the question.

Examples

① $$(2y+1)\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

Use the substitution $w = \frac{dy}{dx}$ to eliminate x from this DE.

Hence find the solution for which $y=4$ and $\frac{dy}{dx} = 6$ when $x = 0$.

$$\frac{dy}{dx} = w \Rightarrow \frac{d^2y}{dx^2} = \frac{dw}{dx}$$ (but this still contains x)

$$= \frac{dw}{dy} \times \frac{dy}{dx}$$

$$= w\frac{dw}{dy}$$

$$\Rightarrow (2y+1)\,w\frac{dw}{dy} = w^2$$

$$\int \frac{1}{w}\,dw = \int \frac{1}{2y+1}\,dy$$

$$\ln w = \frac{1}{2}\ln|2y+1| + C$$

When $y=4$, $w=6 \Rightarrow$

$$\ln 6 = \frac{1}{2}\ln 9 + c$$
$$\ln 2 = c$$

$$\ln w = \frac{1}{2}\ln|2y+1| + \ln 2$$
$$= \ln 2(2y+1)^{1/2}$$
$$w = 2(2y+1)^{1/2}$$

$$\frac{dy}{dx} = 2(2y+1)^{1/2}$$

$$\int (2y+1)^{-1/2}\,dy = \int 2\,dx$$

$$\frac{1}{2}\times\frac{1}{1/2}(2y+1)^{1/2} = 2x + c$$

When $x=0$, $y=4$

$$3 = c$$

$$(2y+1)^{1/2} = 2x+3$$

(The question may state 'expressing y in terms of x' in which case we continue:)

$$2y+1 = (2x+3)^2$$
$$= 4x^2+12x+9$$

$$y = 2x^2 + 6x + 4$$

$$\underline{\underline{y = 2(x+1)(x+2)}}$$

(2) Use the substitution $x = e^t$ to find the general solution of

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = \ln x$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \qquad \text{and} \qquad x = e^t$$

$$\Rightarrow \frac{dx}{dt} = e^t$$

so $\dfrac{dy}{dx} = e^{-t}\dfrac{dy}{dt}$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\
&= \frac{d}{dx}\left(e^{-t}\frac{dy}{dt}\right) \\
&= \frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right) \times \frac{dt}{dx} \\
&= \left[-e^{-t}\frac{dy}{dt} + e^{-t}\frac{d^2y}{dt^2}\right] \times \frac{dt}{dx} \\
&= e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)
\end{aligned}$$

$$e^{2t}e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - 2e^te^{-t}\frac{dy}{dt} + 2y = t$$

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = t$$

By the CF + PI method we find

$$y = Pe^{t} + Qe^{2t} + \frac{1}{2}t + \frac{3}{4}$$

Substitute back $x = e^{t}$:

$$y = Px + Qx^{2} + \frac{1}{2}\ln x + \frac{3}{4}$$