

Matrices

Note Title

09/01/2009

Transformations

A matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be used to represent a linear transformation of the x - y plane to the x' - y' plane.

$$\text{ie/} \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where $\underline{r} = x\underline{i} + y\underline{j}$ is the p.v. of a point R and $\underline{r}' = x'\underline{i} + y'\underline{j}$ is the p.v. of R' , the image of R .

Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ the origin is an invariant point under any matrix transformation.

The point with p.v. \underline{i} or $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is mapped to $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \underline{p}$

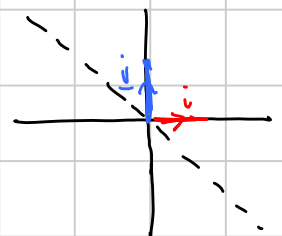
The point with p.v. \underline{j} is mapped to $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} = \underline{q}$

Thus gives a way of finding the matrix for a given transformation.

e.g. a reflection in the line $y = -x$

$$\underline{i} \rightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$
$$\text{and } \underline{j} \rightarrow \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

so the matrix for this transformation is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$



We can view the transformation in two ways :-

either $x\underline{i} + y\underline{j} \rightarrow x'\underline{i} + y'\underline{j}$ where $x' = ax + by$
 $y' = cx + dy$
(the base vectors stay the same, but the components change)

$$\underline{or} \quad x\underline{i} + y\underline{j} \rightarrow x\underline{p} + y\underline{q} \quad \text{ie, } x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix}$$

(the components stay the same, but the base vectors change)
coefficients

eg if $M = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ then a point A with pv $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\text{is transformed to } \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = \underline{5\underline{i} + 5\underline{j}}$$

$$\underline{or} \text{ is transformed to } \underline{2\underline{p} + 1\underline{q}} \left[= 2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \right]$$

[If we view the plane as a grid of unit squares, with sides \underline{i} and \underline{j} , it is transformed to a grid of parallelograms with sides \underline{p} and \underline{q} .]

Singular Matrices

If \underline{q} is a multiple of \underline{p} (eg $M = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$ so that $\underline{q} = -\frac{1}{2}\underline{p}$) then the whole plane is mapped to the line $r = \lambda\underline{p}$. The transformation is MANY-TO-ONE ie, many points in the plane (a whole line of them) are mapped to a single point on the image line. In particular, a whole line of points is mapped to the origin

$$\text{e.g. } \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} 1\lambda \\ 2\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{so under this transformation the line } r = \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ is mapped to the origin.}$$

In this case A is called SINGULAR matrix

Orthogonal Matrices

If \underline{p} and \underline{q} are perpendicular unit vectors (ie, $|\underline{p}| = |\underline{q}| = 1$ and $\underline{p} \cdot \underline{q} = 0$) then M represents a rigid transformation (an isometry) of the plane. In this case M is called an orthogonal matrix. In 2D, orthogonal matrices represent rotations or reflections.

Some common transformations

Enlargement by scale factor k

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

Stretch in x -direction by scale factor k

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$$

Rotation through θ about the origin
(+ve angle \Rightarrow anticlockwise)

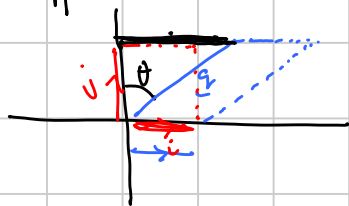
$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

Reflection in the line $y = mx$
where $m = \tan \theta$

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

Shear \parallel to x -axis by angle θ

$$\cos \theta = \frac{1}{|q|}$$



$$\begin{pmatrix} 1 & \tan \theta \\ 0 & 1 \end{pmatrix}$$

Compound Transformations

If transformation P is represented by matrix M and Q by matrix N , then the transformation "P followed by Q" is calculated using

$$N \left(M \begin{pmatrix} x \\ y \end{pmatrix} \right) \quad \text{or} \quad NM \begin{pmatrix} x \\ y \end{pmatrix}$$

ie the single matrix for "P followed by Q" is NM
(note the order!)

Determinant and Area

The DETERMINANT of a 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $|M| = ad - bc$. The transformation M multiplies the area of any shape in the plane by $|M|$. If $|M|$ is negative, the shape is 'turned over' (as in a reflection) by the transformation.

e.g. for a straightforward reflection

$$\begin{vmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{vmatrix} = -\cos^2 2\theta - \sin^2 2\theta = -1$$

If $|M| = 0$, area is destroyed, so M is a SINGULAR matrix which maps the plane to a line (or possibly to a point if $M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$)

Inverse Matrices and Transformations

The inverse of a matrix is written M^{-1} and is defined by

$$M^{-1}M = I \quad \text{where } I \text{ is the identity matrix } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
ie, $\frac{1}{|M|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

e.g. if $M = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$, $|M| = 10$

$$\text{so } M^{-1} = \frac{1}{10} \begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix} \text{ or } \begin{pmatrix} 0.3 & -0.2 \\ -0.1 & 0.4 \end{pmatrix}$$

Note that:

- ① If $|M| = k$, $|M^{-1}| = \frac{1}{k}$ ($|M^{-1}| = |M|^{-1}$)
- ② If $|M| = 0$ M has no inverse (because M is a many-to-one transformation)

③ $(MN)^{-1} = N^{-1}M^{-1}$ ("undo" transformations in reverse order).

The transpose of a matrix

$$\text{If } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad M^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

(rows \leftrightarrow columns, or reflect in leading diagonal)

Note that :-

① If M is orthogonal

$$|\underline{p}| = |\underline{q}| = 1 \quad \text{so} \quad \sqrt{a^2+c^2} = 1 = \sqrt{b^2+d^2}$$

$$\text{also} \quad \underline{p} \cdot \underline{q} = 0 \quad \text{so} \quad ab+cd = 0.$$

$$\begin{aligned} \text{Now consider } M^T M &= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{I} \end{aligned}$$

So for an orthogonal matrix M , $M^{-1} = M^T$

② For any square matrices A and B

$$(AB)^T = B^T A^T$$

③ If $M^T = M$, M is called a SYMMETRIC matrix

$$\text{e.g. } \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}$$

3D and 3x3 Matrices

All the above extends in a natural way to 3D space (and indeed to n -D space)

$\underline{i}, \underline{j}, \underline{k}$ are transformed to $\underline{p}, \underline{q}, \underline{r}$, the columns of M and so a unit cube becomes a parallelepiped.

Singular 3×3 Matrix

If $|M| = 0$, M is singular so 3D space is transformed to :-

- a point (if M is the zero matrix)
- a line (if \underline{p} , \underline{q} and \underline{r} are all multiples of each other)
- a plane (if \underline{p} , \underline{q} and \underline{r} all lie in the same plane)

3×3 Determinants

let $M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

If we choose any element and draw a horizontal and vertical line through it we are left with a 2×2 matrix.

The determinant of this is called the MINOR of the element. So the MINOR of 3 is $4 \times 8 - 5 \times 7 = -3$.

To take account of the 'wrapping round' left to right and top to bottom we attach a sign to this minor according to the following scheme

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

This gives a number called the COFACTOR of the element. So the cofactor of 3 is still -3

But for 2, the MINOR is $4 \times 9 - 6 \times 7 = -6$
but the COFACTOR is $-(-6) = 6$

Now the determinant of M is found by choosing any row or column of M , multiplying each

element in the row column by its cofactor and adding the results.

$$\text{So } \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \text{(using the top row)} \\ = 1 \times (5 \times 9 - 6 \times 8) + 2 \times 6 + 3 \times -3 \\ = -3 + 12 - 9 \\ = 0$$

e.g. $\begin{vmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ 0 & 5 & 2 \end{vmatrix}$ $1 \times 6 - 5$

$$\text{(use the bottom row)} = 0 + 5 \times -(1 - -8) + 2 \times (3 - -4) \\ = -45 + 14 \\ = -31$$

Inverse of a 3x3 matrix

Example

let $M = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & 1 \\ 0 & 4 & 1 \end{pmatrix}$

① Find the determinant: $|M| = 4 \times -(1 - -6) + 1 \times (-3 - -4) \\ = -28 + 1 = -27$

② Find M^T $M^T = \begin{pmatrix} 1 & -2 & 0 \\ 2 & -3 & 4 \\ 3 & 1 & 1 \end{pmatrix}$

③ Replace each element of M^T by its COFACTOR, The resulting matrix is called the ADJOINT of M

$$\text{adj}(M) = \begin{pmatrix} -7 & 10 & 11 \\ 2 & 1 & -7 \\ -8 & -4 & 1 \end{pmatrix}$$

(4) Divide $\text{adj}(M)$ by $|M|$ to get M^{-1}

$$M^{-1} = \frac{1}{27} \begin{pmatrix} +7 & -10 & -11 \\ -2 & -1 & +7 \\ +8 & 4 & -1 \end{pmatrix}$$

(5) Check!

$$\begin{aligned} M^{-1}M &= \frac{1}{27} \begin{pmatrix} 7 & -10 & -11 \\ -2 & -1 & 7 \\ 8 & 4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & 1 \\ 0 & 4 & 1 \end{pmatrix} \\ &= \frac{1}{27} \begin{pmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{pmatrix} = I \end{aligned}$$

Image of a line and a plane

Example Find the image of

(a) the line $\frac{x-1}{2} = \frac{z-1}{3}$; $y=2$

(b) the plane $x-2y-z=2$

under the transformation $M = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & 1 \\ 0 & 4 & 1 \end{pmatrix}$

(a) this is best handled with the line in vector form

$$\frac{x-1}{2} = \frac{z-1}{3} = \lambda \Rightarrow \begin{aligned} x &= 1 + 2\lambda \\ y &= 2 \\ z &= 1 + 3\lambda \end{aligned}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

$\vec{r} = \vec{a} + \lambda \vec{v}$

$$\begin{aligned} \text{Hence } M\vec{r} &= M \left[\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \right] \\ &= \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & 1 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & 1 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \end{aligned}$$

$$\underline{r'} = \begin{pmatrix} 8 \\ -7 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 11 \\ -1 \\ 3 \end{pmatrix}$$

(b) This is best tackled using the inverse matrix.

Let the new coordinate system be $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = M \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Then $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{27} \begin{pmatrix} 7 & -10 & -11 \\ -2 & -1 & 7 \\ 8 & 4 & -1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$x = \frac{1}{27} (7x' - 10y' - 11z')$$

$$y = \frac{1}{27} (-2x' - y' + 7z')$$

$$z = \frac{1}{27} (8x' + 4y' - z')$$

But we want to consider points on the plane

$$x - 2y - z = 2$$

$$\frac{1}{27} (7x' - 10y' - 11z' - 2(-2x' - y' + 7z') - (8x' + 4y' - z')) = 2$$

$$3x' - 12y' - 24z' = 54$$

$$x' - 4y' - 8z' = 18 \quad \left[\text{or } r' \cdot (\underline{i} - 4\underline{j} - 8\underline{k}) = 18 \right]$$

p74 Ex 3B Q 4 (some until confident)

p165 Review Ex. Q 3, 9, 15, 27, 29, 41, 44, 69

Eigenvalues and Eigenvectors

Consider the product

$$\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \text{ which is just } 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

ie, the effect of multiplying $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ by the matrix $M = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$ is the same as multiplying by the scalar 2.

We say that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an EIGENVECTOR of M , with EIGENVALUE 2.

Eigenvectors are not unique - any vector $k \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is also an eigenvector of M with eigenvalue 2.

So the line $r = \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (through the origin) is mapped to $r' = Mr$
ie, $r' = 2\lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

which is the same line 'stretched' by a factor 2.

We say that $r = \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an INVARIANT LINE under the transformation M .

If the eigenvalue were 1, ie, $M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ for some vector $\begin{pmatrix} x \\ y \end{pmatrix}$, the line is called a LINE OF INVARIANT POINTS.

To find the eigenvalues and eigenvectors of a given matrix, we use the following theory :-

Given M , we need λ and \underline{v} such that

$$M \underline{v} = \lambda \underline{v}$$

$$\Rightarrow M \underline{v} - \lambda \underline{v} = \underline{0} \text{ or } \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow M \underline{v} - \lambda \underline{I} \underline{v} = \underline{0}$$

$$\Rightarrow (M - \lambda \underline{I}) \underline{v} = \underline{0}$$

This statement says that the matrix $M - \lambda \underline{I}$ transforms the non-zero vector \underline{v} to the origin $\underline{0}$.

This is only possible if $M - \lambda \underline{I}$ is a SINGULAR matrix ie if

$$\boxed{|M - \lambda \underline{I}| = 0}$$

This is called the 'characteristic equation' of M and is the starting point for finding the eigenvalues.

Example 1

(a) Find the values of $M = \frac{1}{10} \begin{pmatrix} 11 & 3 \\ 3 & 19 \end{pmatrix}$ and find a normalized (ie unit) vector associated with each value

(b) Hence describe geometrically the transformation represented by this matrix.

$$(a) \quad |M - \lambda \underline{I}| = 0$$

$$\left| \begin{pmatrix} \frac{11}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{19}{10} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} \frac{11}{10} - \lambda & \frac{3}{10} \\ \frac{3}{10} & \frac{19}{10} - \lambda \end{vmatrix} = 0$$

$$\left(\frac{11}{10} - \lambda\right)\left(\frac{19}{10} - \lambda\right) - \frac{9}{100} = 0$$

$$\frac{209}{100} - \frac{30}{10}\lambda + \lambda^2 - \frac{9}{100} = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 2)(\lambda - 1) = 0$$

$$\lambda = 2 \text{ or } \lambda = 1$$

Eigenvectors for $\lambda = 2$:

$$\frac{1}{10} \begin{pmatrix} 11 & 3 \\ 3 & 19 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left. \begin{aligned} 11x + 3y &= 20x \\ 3x + 19y &= 20y \end{aligned} \right\}$$

$$\left. \begin{aligned} -9x + 3y &= 0 \\ 3x - y &= 0 \end{aligned} \right\} \text{ these are the same equation, as we would expect}$$

$$\text{let } x = k_1 \Rightarrow y = 3k_1$$

$$\text{So the eigenvectors are } \begin{pmatrix} k_1 \\ 3k_1 \end{pmatrix} \text{ or } k_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\text{The unit eigenvector is } \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (= \underline{b})$$

$$\left[\begin{array}{l} \text{The vector eqn of the invariant line is } \underline{r} = \lambda \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ \text{The Cartesian eqn is } y = 3x \end{array} \right]$$

For $\lambda = 1$:

$$\frac{1}{10} \begin{pmatrix} 11 & 3 \\ 3 & 19 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left. \begin{aligned} 11x + 3y &= 10x \\ 3x + 19y &= 10y \end{aligned} \right\}$$

$$\left. \begin{aligned} x + 3y &= 0 \\ 3x + 9y &= 0 \end{aligned} \right\}$$

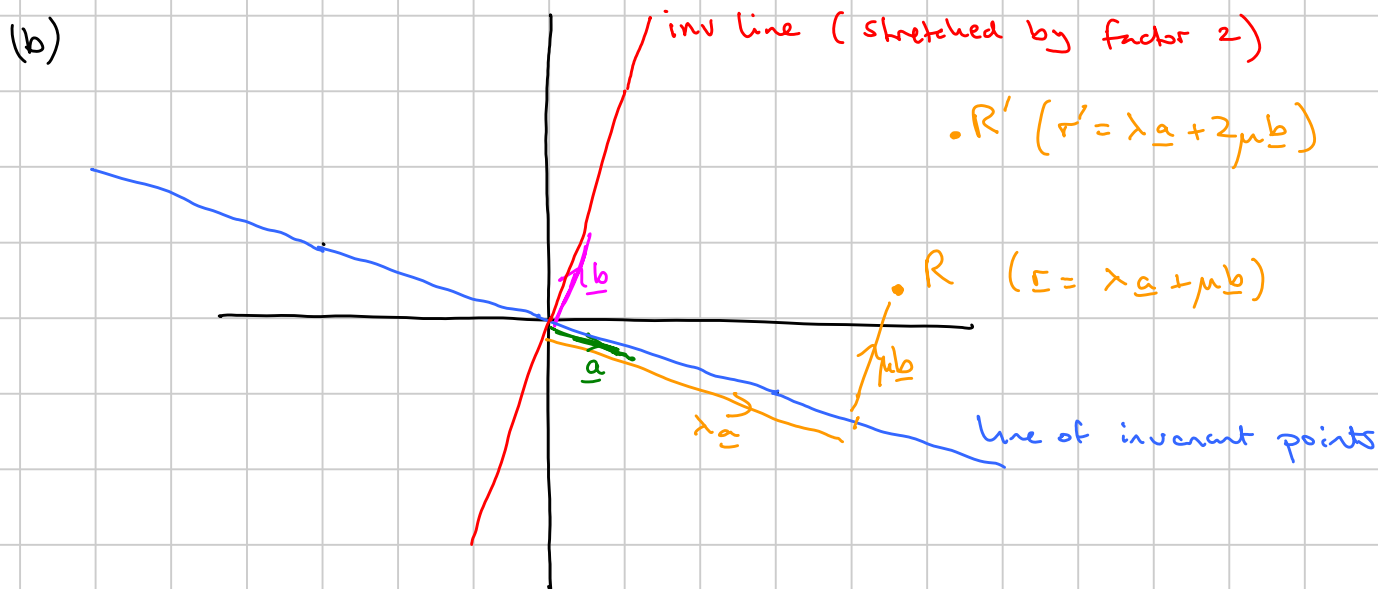
$$\text{let } y = k_2 \Rightarrow x = -3k_2$$

$$\text{Eigenvectors are } k_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\text{Unit eigenvector is } \frac{1}{\sqrt{10}} \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\text{let } \underline{a} = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

So $y = -\frac{1}{3}x$ is a line of invariant points



The pv of any point \underline{r} in the plane can be written as

$$\underline{r} = \lambda \underline{a} + \mu \underline{b}$$

This is transformed by M to

$$\begin{aligned} \underline{r}' &= M \underline{r} \\ &= M (\lambda \underline{a} + \mu \underline{b}) \\ &= \lambda M \underline{a} + \mu M \underline{b} \\ &= \lambda \underline{a} + 2\mu \underline{b} \end{aligned}$$

The transformation is a stretch by scale factor 2 perpendicular to the line of invariant points $y = -\frac{1}{3}x$.

Example 2 Given that $\lambda = 2$ is an eigenvalue of

$$M = \frac{1}{3} \begin{pmatrix} 5 & -2 & 2 \\ -2 & 2 & 4 \\ 2 & 4 & 2 \end{pmatrix}, \text{ find all the ev values, and find}$$

a set of 3 mutually perpendicular e vectors for M .

$$|M - \lambda I| = 0$$

$$\begin{vmatrix} 5/3 - \lambda & -2/3 & 2/3 \\ -2/3 & 2/3 - \lambda & 4/3 \\ 2/3 & 4/3 & 2/3 - \lambda \end{vmatrix} = 0$$

$$\left(\frac{5}{3} - \lambda\right) \left[\left(\frac{2}{3} - \lambda\right) \left(\frac{2}{3} - \lambda\right) - \frac{16}{9} \right] + \frac{2}{3} \left[-\frac{2}{3} \left(\frac{2}{3} - \lambda\right) - \frac{8}{9} \right] + \frac{2}{3} \left[-\frac{8}{9} - \frac{2}{3} \left(\frac{2}{3} - \lambda\right) \right] = 0$$

$$\left(\frac{5}{3} - \lambda\right) \left[\frac{4}{9} - \frac{4}{3}\lambda + \lambda^2 - \frac{16}{9} \right] - \frac{8}{27} + \frac{4}{9}\lambda - \frac{16}{27} - \frac{16}{27} - \frac{8}{27} + \frac{4}{9}\lambda = 0$$

$$-\frac{60}{27} + \frac{12}{9}\lambda - \frac{20}{9}\lambda + \frac{4}{3}\lambda^2 + \frac{5}{3}\lambda^2 - \lambda^3 - \frac{48}{27} + \frac{8}{9}\lambda = 0$$

$$-\frac{108}{27} + \frac{9}{3}\lambda^2 - \lambda^3 = 0$$

$$\lambda^3 - 3\lambda^2 + 4 = 0$$

$\lambda = 2$ is a root $\Rightarrow \lambda - 2$ is a factor

$$\begin{array}{r} \lambda^2 - \lambda - 2 \\ \lambda - 2 \overline{) \lambda^3 - 3\lambda^2 + 0\lambda + 4} \\ \underline{\lambda^3 - 2\lambda^2} \\ -\lambda^2 + 0\lambda \\ \underline{-\lambda^2 + 2\lambda} \\ -2\lambda + 4 \\ \underline{-2\lambda + 4} \\ 0 \end{array}$$

$$\lambda^3 - 3\lambda^2 + 4 = 0$$

$$(\lambda - 2)(\lambda - 2)(\lambda + 1) = 0$$

Eigenvalues are $\lambda = 2$ (repeated)
 $\lambda = -1$

Eigenvectors for $\lambda = -1$

$$\frac{1}{3} \begin{pmatrix} 5 & -2 & 2 \\ -2 & 2 & 4 \\ 2 & 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = - \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\left. \begin{array}{l} 5x - 2y + 2z = -3x \\ -2x + 2y + 4z = -3y \\ 2x + 4y + 2z = -3z \end{array} \right\}$$

$$\left. \begin{aligned} 8x - 2y + 2z &= 0 & \textcircled{1} \\ -2x + 5y + 4z &= 0 & \textcircled{2} \\ 2x + 4y + 5z &= 0 & \textcircled{3} \end{aligned} \right\}$$

Eliminate z :

$$\begin{aligned} 2 \times \textcircled{1} - \textcircled{2} &\Rightarrow 18x - 9y = 0 \\ &\Rightarrow 2x - y = 0 \end{aligned}$$

let $x = k_1 \Rightarrow y = 2k_1$

Subst into $\textcircled{1} \Rightarrow 8k_1 - 4k_1 + 2z = 0$
 $z = -2k_1$

Eigenvectors are $k_1 \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

Unit eigenvector is $\frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

Eigenvectors for $\lambda = 2$:

$$\frac{1}{3} \begin{pmatrix} 5 & -2 & 2 \\ -2 & 2 & 4 \\ 2 & 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\left. \begin{aligned} 5x - 2y + 2z &= 6x \\ -2x + 2y + 4z &= 6y \\ 2x + 4y + 2z &= 6z \end{aligned} \right\}$$

$$\left. \begin{aligned} -x - 2y + 2z &= 0 \\ -2x - 4y + 4z &= 0 \\ 2x + 4y - 4z &= 0 \end{aligned} \right\}$$

These are all the same equation, which is therefore the equation of a plane, which is stretched by scale factor 2 by M

We need two parameters:

$$\text{let } y = k_2 \text{ and } z = k_3 \Rightarrow x = -2k_2 + 2k_3$$

Eigenvectors for $\lambda = 2$ are of the form

$$\begin{pmatrix} -2k_2 + 2k_3 \\ k_2 \\ k_3 \end{pmatrix}$$

$$\text{or } k_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + k_3 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

[So a parametric vector eqn of this plane is

$$\underline{r} = \underline{0} + k_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + k_3 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

]

$$\text{Now } k_1 \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -2k_2 + 2k_3 \\ k_2 \\ k_3 \end{pmatrix} = k_1 \left[(-2k_2 + 2k_3) + 2k_2 - 2k_3 \right] \\ = 0$$

So the eigenvectors for $\lambda = -1$ are perpendicular to the plane of eigenvectors for $\lambda = 2$.

So we need 2 vectors in the plane perpendicular to each other.

Choose $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ as one of them.

$$\text{Then we need } \begin{pmatrix} -2k_2 + 2k_3 \\ k_2 \\ k_3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 0$$

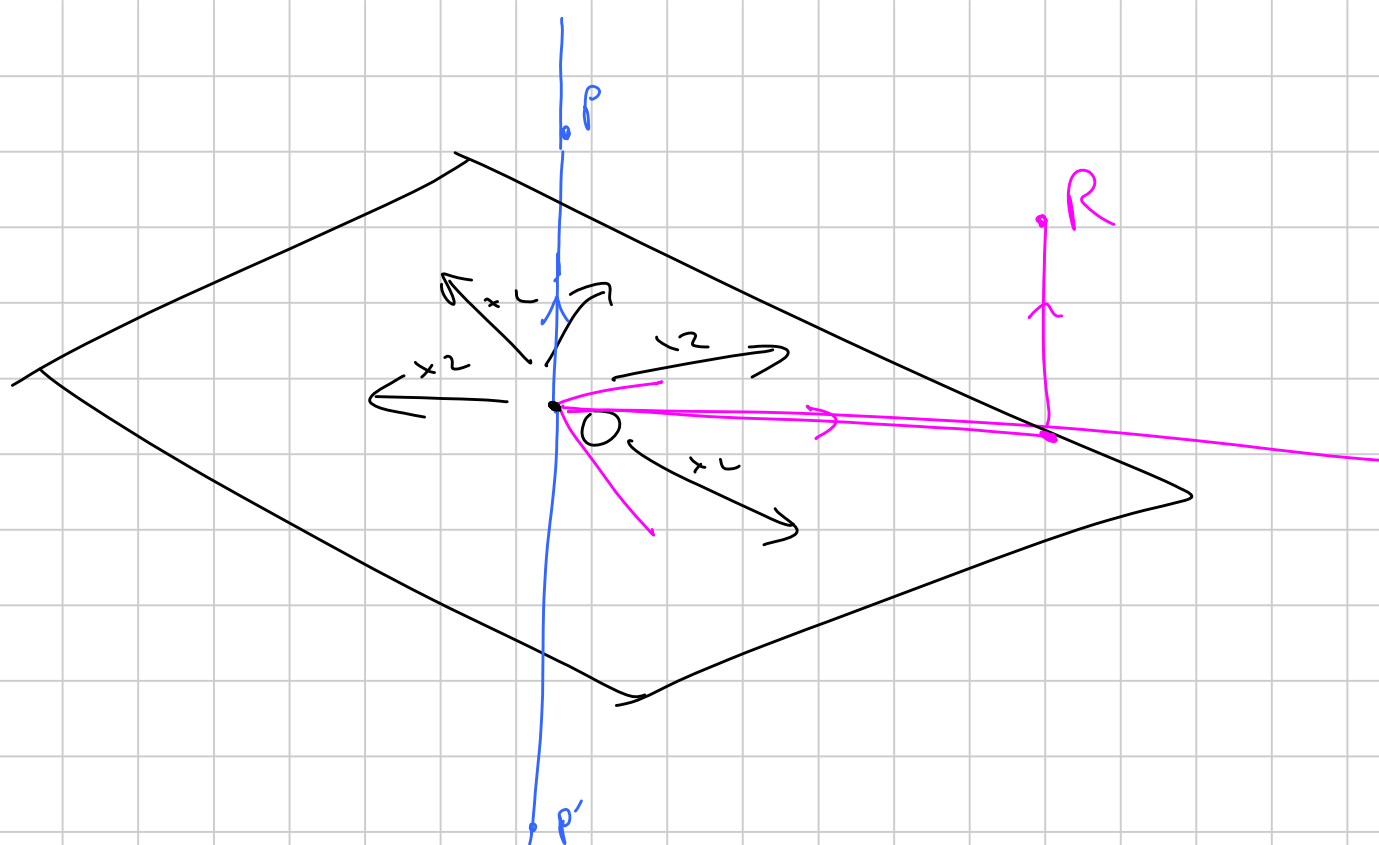
$$-4k_2 + 4k_3 + k_3 = 0$$

$$-4k_2 + 5k_3 = 0$$

$$\text{e.g. } k_2 = 5, k_3 = 4.$$

So $\begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix}$ is perpendicular to $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

3 mutually perpendicular eigenvectors are $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix}$
 $(\lambda = -1)$ $\lambda = 2$



p 105 \mathbb{E}^3 3D Q 1 (a)(d)
 2 (a)(b)(c)

(d) Diagonalisation

A DIAGONAL matrix has the form $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$. It is

very easy to interpret geometrically.

Example 1 (cont)

We saw above that the matrix $M = \frac{1}{10} \begin{pmatrix} 11 & 3 \\ 3 & 19 \end{pmatrix}$ represents a stretch by s.f 2 in the direction of $\underline{b} = \begin{pmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix}$.

In other words, if the axes were in the direction of \underline{a} and \underline{b} , the matrix for the transformation would be

$\sqrt{\lambda} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, a diagonal matrix whose entries on

the leading diagonal are the eigenvalues of M .

We could perform the transformation M in 3 stages

- ① Rotate the plane so that $\underline{a} \rightarrow \underline{i}$ and $\underline{b} \rightarrow \underline{j}$, using a matrix P^{-1} .
- ② Stretch in the y -direction using $\sqrt{\lambda}$.
- ③ Rotate back so that $\underline{i} \rightarrow \underline{a}$ and $\underline{j} \rightarrow \underline{b}$ using the matrix P .

What is the matrix P ? Its first column is \underline{a} and its second column is \underline{b} .

So the columns of P are the eigenvectors of M .

ie,
$$P = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}$$

We can generalise this, and extend it to 3D:

If M is a 3×3 matrix, then provided M has 3 eigenvectors \underline{a} , \underline{b} and \underline{c} which are not all in the same plane (they need not be mutually perpendicular)

then we can perform M in stages:-

- ① Take \underline{a} , \underline{b} , \underline{c} to \underline{i} , \underline{j} , \underline{k} using P^{-1}
- ② Stretch the axes using the eigenvalues of M (if an eigenvalue is $-ve$, the axis is also reflected), by using a diagonal matrix $\sqrt{\lambda}$
- ③ Take \underline{i} , \underline{j} , \underline{k} back to \underline{a} , \underline{b} , \underline{c} using P where P 's columns are the eigenvectors of M in the order corresponding to the eigenvalues in $\sqrt{\lambda}$

So
$$M = P \sqrt{\lambda} P^{-1}$$

$$\text{or } P^{-1}M = P^{-1}P \Lambda P^{-1} = \Lambda P^{-1}$$

$$P^{-1}MP = \Lambda P^{-1}P$$

$$P^{-1}MP = \Lambda$$

(we say that 'P diagonalises M')

Not all matrices are diagonalisable.

SYMMETRIC matrices have the following property:-

We can always find a set of 3 mutually perpendicular unit eigenvectors a, b, c , so can be diagonalised using a matrix P which is **ORTHOGONAL**. In this case P^{-1} is the same as P^T ,

so

$$P^T M P = \Lambda$$

Example 2 (cont)

For the matrix $M = \frac{1}{3} \begin{pmatrix} 5 & -2 & 2 \\ -2 & 2 & 4 \\ 2 & 4 & 2 \end{pmatrix}$ find a matrix P and a matrix Λ such that $P^T M P = \Lambda$.

In order for P to be orthogonal, we make the mutually perpendicular vectors found above into unit vectors and use them as the columns of P .

$$\text{So } \Lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } P = \begin{pmatrix} 1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & 4/\sqrt{45} \end{pmatrix}$$

p106 Ex 3D Q 3 (a)(e)(f)